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Bochner Theorems for Hypergroups and Their Applications to Orthogonal Polynomial Expansions

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INTRODUCTION

Positive definite functions associated with the ultraspherical polynomials were studied by Schoenberg [24], Kennedy [18], and Bingham [5]. Generalizing Bochner's theorem, it was shown that positive definite functions associated with ultraspherical polynomials are exactly the absolutely convergent ultraspherical expansions having nonnegative coefficients, see [18, Theorem 3.1]. In [25, 26] Schwartz investigated absolutely convergent nonnegative expansions in certain general orthogonal polynomials. Here we shall prove a Bochner theorem in the setting of commutative hypergroups. An application of this theorem characterizes the expansions studied by Schwartz as exactly the positive definite functions. For that, positive definiteness is defined in a natural way. The correspondence between hypergroups and certain orthogonal polynomial sequences was recently established in [19].

The theory of hypergroups has been developed in [11, 17, 27] and has received a good deal of attention from harmonic analysts. Hypergroups arise as double coset spaces of locally compact groups. As yet mentioned, certain orthogonal polynomial sequences bear a hypergroup structure, too. Our main reference for hypergroups will be [17].

Throughout this note K will be a commutative hypergroup. Denote by

$$\hat{K} = \{\alpha: K \to \mathbb{C}: \alpha \text{ continuous, bounded, } \alpha(e) = 1,$$
$$p_x * p_y(\alpha) = \alpha(x)\,\alpha(y),\ \alpha(\bar{x}) = \overline{\alpha(x)}\},$$

the character space of K. Equipped with the topology of uniform

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convergence on compacta, $\hat{K}$ is a locally compact Hausdorff space. For each $\mu \in M(K)$ the Fourier transform is given by

$$\hat{\mu}(\alpha) = \int \overline{\alpha(x)}\, d\mu(x).$$

For $\mu \in M(\hat{K})$ the inverse Fourier transform is defined by

$$\check{\mu}(x) = \int \alpha(x)\, d\mu(\alpha).$$

We shall say that $\hat{K}$ is a hypergroup with respect to pointwise multiplication, if for $\alpha, \beta \in \hat{K}$ there exists a probability measure $p_\alpha * p_\beta \in M(\hat{K})$ such that

$$\alpha(x)\beta(x) = (p_\alpha * p_\beta)\check{}\,(x)$$

for each $x \in K$, and $\hat{K}$ is a hypergroup with this convolution and the complex conjugation as involution and one as unit. In general the dual $\hat{K}$ of a commutative hypergroup K is not a hypergroup with respect to pointwise multiplication, see [9, Example 4.8; 17, Example 9.1c], or Examples 2f and 2g in Section 2. If $\hat{K}$ is a dual hypergroup, then $K \subseteq (\hat{K})\hat{}$ in a natural manner, [17, Theorem 12.4B]. If in addition $K = (\hat{K})\hat{}$ holds, then [17, Theorem 12.3B] shows that the Fourier transforms $\hat{\mu}$ of positive measures $\mu \in M(K)$ are exactly the bounded positive definite functions on $\hat{K}$.

The aim of this treatise is to characterise the Fourier transforms $\hat{\mu}$ of positive measures $\mu \in M(K)$, where K is an arbitrary commutative hypergroup without any assumptions on the dual $\hat{K}$. Further we describe the applications for orthogonal polynomial expansions.

1. Bochner Theorems

Recall that K denotes a commutative hypergroup. A continuous bounded function $\varphi \in C(\hat{K})$ on $\hat{K}$ is called *strongly positive definite* if for any $\mu \in M(\hat{K})$ with $\check{\mu} \geqslant 0$, the inequality $\int \varphi(\alpha)\, d\mu(\alpha) \geqslant 0$ holds. If φ is strongly positive definite, then $\varphi(1) = \int \varphi(\alpha)\, dp_1(\alpha) \geqslant 0$. The set of all strongly positive definite functions on $\hat{K}$ is denoted by $\mathrm{SP}(\hat{K})$. A measure $\mu \in M(\hat{K})$, such that $\check{\mu} \geqslant 0$ holds, we shall call positive definite. The set of all positive definite measures is denoted by $\mathrm{PM}(\hat{K})$.

Proposition 1. *Let φ be in $\mathrm{SP}(\hat{K})$.*

(a) *If $\mu \in \mathrm{PM}(\hat{K})$, then $(\varphi\mu)\check{} \geqslant 0$.*

(b) *If $\mu \in M(\hat{K})$ such that $\check{\mu}$ is real valued, then $\int \varphi(\alpha)\, d\mu(\alpha)$ is real.*

(c) *The equality* $\varphi(\bar{\alpha}) = \overline{\varphi(\alpha)}$ *holds, and* $\bar{\varphi}$ *and* $\operatorname{Re}\varphi$ *are also in* $\mathrm{SP}(\hat{K})$.

(d) *If* ψ *is a further element of* $\mathrm{SP}(\hat{K})$, *then* $\varphi\psi$ *and* $c_1\varphi + c_2\psi$ *are in* $\mathrm{SP}(\hat{K})$ *for* $c_1, c_2 \geqslant 0$.

Proof. (a) If $\mu \in \mathrm{PM}(\hat{K})$, then $(\widehat{p_x\mu})\check{}\,(y) = \int \check{\mu}(z)\, dp_{\bar{x}} * p_y(z) \geqslant 0$ for each $x, y \in K$. Thus $(\varphi\mu)\check{}\,(\bar{x}) = \int \varphi(\alpha)\, d(\widehat{p_x\mu})(\alpha) \geqslant 0$ for any $x \in K$.

(b) Let $\check{\mu}(x) \in \mathbb{R}$ for any $x \in K$. Then $\|\check{\mu}\|_u p_1 + \mu \in M(\hat{K})$ and $(\|\check{\mu}\|_u p_1 + \mu)\check{}\, \geqslant 0$. Thus $\|\check{\mu}\|_u \varphi(1) + \int \varphi(\alpha)\, d\mu(\alpha) \geqslant 0$. In particular $\int \varphi(\alpha)\, d\mu(\alpha) \in \mathbb{R}$.

(c) Since $(p_\alpha + p_{\bar{\alpha}})\check{}\, = 2 \operatorname{Re}\alpha$, we have $\varphi(\alpha) + \varphi(\bar{\alpha}) \in \mathbb{R}$ by (b). Since $(p_\alpha - p_{\bar{\alpha}})\check{}\,/i = 2 \operatorname{Im}\alpha$, we see that $(\varphi(\alpha) - \varphi(\bar{\alpha}))/i \in \mathbb{R}$. Hence $\varphi(\bar{\alpha}) = \overline{\varphi(\alpha)}$. Further by $(\mu^*)\check{}\, = \bar{\check{\mu}}$, [17, 12.1F], we have $\int \overline{\varphi(\alpha)}\, d\mu(\alpha) = \int \varphi(\bar{\alpha})\, d\mu(\alpha) = \overline{\int \varphi(\alpha)\, d^*(\alpha)} \geqslant 0$. Thus $\bar{\varphi} \in \mathrm{SP}(\hat{K})$, and then $\operatorname{Re}\varphi \in \mathrm{SP}(\hat{K})$ holds.

(d) The product $\varphi\psi$ is strongly positive definite by (a).

Denote by m the Haar measure on K, see [28], and imbed $L^1(K) = L^1(m)$ into $M(K)$ as usual. Levitan's theorem, [17, 7.3I], yields a nonnegative measure π on $\hat{K}$, the Plancherel measure, such that

$$\int |f(x)|^2\, dm(x) = \int |\hat{f}(\alpha)|^2\, d\pi(\alpha) \qquad \text{for} \quad f \in L^1(K) \cap L^2(K).$$

Also we imbed $L^1(\hat{K}) = L^1(\pi)$ into $M(\hat{K})$. For a locally compact space X denote $C_{00}(X)$ (resp. $C_0(X)$ the space of all continuous functions having compact support (resp. vanishing at infinity).

PROPOSITION 2. $(C_{00}(\hat{K}))\check{}$ *is a* sup-norm *dense subspace of* $C_0(K)$.

Proof. By [6, Theorem 2.4.1] we know that $(C_{00}(\hat{K}))\check{}$ is a subspace of $C_0(K)$. Assume that $(C_{00}(\hat{K}))\check{}$ is not sup-norm dense in $C_0(K)$. From the Hahn–Banach theorem and Riesz' representation theorem there exists a $\mu \in M(K)$, $\mu \neq 0$, such that $\int \check{h}(x)\, d\mu(x) = 0$ for any $h \in C_{00}(\hat{K})$. Thus $0 = \iint h(\alpha)\, \alpha(x)\, d\pi(\alpha)\, d\mu(x) = \int h(\alpha)\, \bar{\hat{\mu}}(\alpha)\, d\pi(\alpha)$ for any $h \in C_{00}(\hat{K})$. By [17, Lemma 12.2B] we have $\mu = 0$, a contradiction.

THEOREM 1. *Let* φ *be in* $\mathrm{SP}(\hat{K})$. *Then there exists a unique positive measure* $\nu \in M(K)$ *such that* $\varphi | \operatorname{supp}\pi = \hat{\nu} | \operatorname{supp}\pi$. *Conversely* $\hat{\nu}$ *is strongly positive definite for each positive measure* $\nu \in M(K)$.

Proof. At first let $\nu \in M(K)$ be a positive measure. If $\mu \in \mathrm{PM}(\hat{K})$, then $\int \hat{\nu}(\alpha)\, d\mu(\alpha) = \int \check{\mu}(\bar{x})\, d\nu(x) \geqslant 0$, i.e., $\hat{\nu} \in \mathrm{SP}(\hat{K})$. Now assume that $\varphi \in \mathrm{SP}(\hat{K})$. If $\mu \in M(\hat{K})$ with $\check{\mu}$ being real valued, then $(\|\check{\mu}\|_u\, p_1 \pm \mu)\check{}\, \geqslant 0$ holds. Thus $\|\check{\mu}\|_u\, \varphi(1) \pm \int \varphi(\alpha)\, d\mu(\alpha) \geqslant 0$, and then $|\int \varphi(\alpha)\, d\mu(\alpha)| \leqslant \varphi(1)\, \|\check{\mu}\|_u$. For an arbitrary $\mu \in M(\hat{K})$ denote by $\mu_1 = (\mu + \mu^*)/2$, $\mu_2 = (\mu - \mu^*)/(2i)$. Then

$\check{\mu}_1 = \operatorname{Re} \check{\mu}$, $\check{\mu}_2 = \operatorname{Im} \check{\mu}$. Hence $|\int \varphi(\alpha)\, d\mu(\alpha)| \leqslant \varphi(1) \|\check{\mu}_1\|_u + \varphi(1) \|\check{\mu}_2\|_u \leqslant 2\varphi(1) \|\check{\mu}\|_u$. Thus $\phi: M(\hat{K})\check{\ } \to \mathbb{C}$, $\phi(\check{\mu}) = \int \varphi(\alpha)\, d\mu(\alpha)$ is a sup-norm continuous linear functional. Denote by ϕ_0 the extension to $C_0(K)$ of $\phi \,|\, (C_{00}(\hat{K}))\check{\ }$. There exists a measure $\nu \in M(K)$ such that

$$\int \varphi(\alpha)\, h(\alpha)\, d\pi(\alpha) = \phi_0(\check{h}) = \int \check{h}(x)\, d\nu(x)$$

for each $h \in C_{00}(\hat{K})$. Further ν is a positive measure. In fact, given $f \in C_0(K)$, $f \geqslant 0$ and $\varepsilon > 0$, Proposition 2 yields a function $h \in C_{00}(\hat{K})$ such that $\|f - \check{h}\|_u < \varepsilon$. We may assume that $\check{h}$ is real valued. Define $\mu = \varepsilon p_1 + h \in M(\hat{K})$. Then $\check{\mu} \geqslant 0$, and hence $\phi(\check{\mu}) \geqslant 0$. Since $|\int f(x)\, d\nu(x) - \phi(\check{\mu})| \leqslant 4\varphi(1)\, \varepsilon$, we see that $\int f(x)\, d\nu(x) \geqslant 0$. Thus ν and also $\bar{\nu}$ are positive. If $h \in C_{00}(\hat{K})$, then

$$\int \varphi(\alpha)\, h(\alpha)\, d\pi(\alpha) = \int \check{h}(x)\, d\nu(x) = \int h(\alpha)\, \hat{\bar{\nu}}(\alpha)\, d\pi(\alpha).$$

The continuity of φ and $\hat{\bar{\nu}}$ implies, that $\varphi = \hat{\bar{\nu}}$ on $\operatorname{supp} \pi$. Finally the uniqueness of ν follows by [17, Lemma 12.2B].

Corollary 1. (a) *If* $\varphi \in \mathrm{SP}(\hat{K})$, *then* $|\varphi(\alpha)| \leqslant \varphi(1)$ *for each* $\alpha \in \operatorname{supp} \pi$.

(b) *Assume that* $\operatorname{supp} \pi = \hat{K}$. *If* (φ_n) *is a sequence of functions of* $\mathrm{SP}(\hat{K})$ *such that* φ_n *converges uniformly on compact subsets of* $\hat{K}$ *to a continuous function* φ, *then* φ *is strongly positive definite.*

Proof. (a) follows by Theorem 1.

(b) By (a) and $\varphi_n(1) \to \varphi(1)$ there exists a constant $M \geqslant 0$ such that $|\varphi_n(\alpha)| \leqslant M$ for each $n \in \mathbb{N}$ and each $\alpha \in \hat{K}$. Let $\mu \in \mathrm{PM}(\hat{K})$ and $\varepsilon > 0$. Choose a compact subset $C \subseteq \hat{K}$ such that $|\mu| \,|\, \hat{K} \setminus C < \varepsilon$. Then

$$\left| \int \varphi(\alpha)\, d\mu(\alpha) - \int \varphi_n(\alpha)\, d\mu(\alpha) \right| \leqslant 2M\varepsilon + \int_C |\varphi(\alpha) - \varphi_n(\alpha)|\, d\,|\mu|(\alpha).$$

Now it is obvious that $\int \varphi(\alpha)\, d\mu(\alpha) \geqslant 0$.

Equip the space $M(K)$ with the weak topology, i.e., the topology defined by the duality $\langle M(K), C^b(K) \rangle$, where $C^b(K)$ is the space of all bounded continuous functions, see, e.g., [16, p. 24].

Corollary 2. *Assume that* $\operatorname{supp} \pi = \hat{K}$. *The Fourier transformation* $\hat{}: M^+(K) \to \mathrm{SP}(\hat{K})$ *defined on the set* $M^+(K)$ *of all positive measures of* $M(K)$ *is a homeomorphism, where* $M^+(K)$ *bears the weak topology and* $\mathrm{SP}(\hat{K})$ *the topology of uniform convergence on compact subsets of* $\hat{K}$.

Proof. One has to make minor modifications of the proof in the group case. First, assume that (ν_κ) is a net with $\nu_\kappa \in M^+(K)$, and $\nu \in M^+(K)$ such that $\nu_\kappa \to \nu$ weakly. Let $\alpha_0 \in \hat{K}$ and $\varepsilon > 0$. Similar arguments as (e.g., in the proof of Theorem 3.13 of [4, p. 15]) yield an index κ_0 and a neighbourhood V_{α_0} of α_0 such that $|\hat{\nu}_\kappa(\alpha_0) - \hat{\nu}_\kappa(\alpha)| < \varepsilon$ for each $\alpha \in V_{\alpha_0}, \kappa \geqslant \kappa_0$. But now it is routine to prove that $\widehat{\nu_\kappa} \to \hat{\nu}$ on compact subsets of $\hat{K}$. Conversely, assume that $\widehat{\nu_\kappa}$ tends to $\hat{\nu}$ in the topology of compact convergence. Using Proposition 2 the arguments of [4, p. 16] yield that ν_α tends to ν vaguely. By [16, Theorem 1.1.9] we have $\nu_\alpha \to \nu$ weakly.

Remark. In general $\operatorname{supp} \pi$ is a proper subset of $\hat{K}$. But if K is compact, then $\operatorname{supp} \pi = \hat{K}$. In fact let $\alpha \in \hat{K}$. By [17, 7.3I] we see that $0 < \int |\alpha(x)|^2 \, dm(x) = \int |\hat{\alpha}(\beta)|^2 \, d\pi(\beta)$. But $\hat{\alpha}(\beta) = \int \alpha(x) \overline{\beta(x)} \, dm(x) = 0$ for $\alpha \neq \beta$, [27, Theorem II.2.2]. Thus $\alpha \in \operatorname{supp} \pi$.

Corollary 3. *Let* $\varphi \in \mathrm{SP}(\hat{K}) \cap L^1(\hat{K})$. *Then* $\check{\varphi}$ *is nonnegative,* $\check{\varphi} \in L^1(K)$, and $(\check{\varphi})\hat{}\,(\alpha) = \varphi(\alpha)$ *for each* $\alpha \in \operatorname{supp} \pi$.

Proof. By Theorem 1 we may write $\varphi(\alpha) = \hat{\nu}(\alpha)$ for each $\alpha \in \operatorname{supp} \pi$, where $\nu \in M^+(K)$. Thus $\hat{\nu}\pi = \varphi\pi \in M(\hat{K})$. By [17, Lemma 12.2B] we see that $\nu = \check{\varphi} m$; i.e., $\check{\varphi} \in L^1(K)$ and $\check{\varphi} \geqslant 0$. Again [17, 12.2B] yields that $\varphi\pi = (\check{\varphi} m)\hat{}\,\pi$. Since $(\check{\varphi})\hat{}$ and φ are continuous functions, $(\check{\varphi})\hat{}\,(\alpha) = \varphi(\alpha)$ holds for every $\alpha \in \operatorname{supp} \pi$.

If $\hat{K}$ is a hypergroup with respect to pointwise multiplication one can consider the relation of strongly positive definite functions to bounded positive definite functions on $\hat{K}$. A continuous function $\varphi \in C(\hat{K})$ is called positive definite, if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \bar{\lambda}_j p_{\alpha_i} * p_{\overline{\alpha_j}}(\varphi) \geqslant 0$$

for any choice of $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $\alpha_1, \ldots, \alpha_n \in \hat{K}$, compare [17, 11.1].

Proposition 3. *Assume that* $\hat{K}$ *is a hypergroup with respect to pointwise multiplication. If* φ *is in* $\mathrm{SP}(\hat{K})$, *then* $\widehat{p_x}\,\varphi$ *is a bounded positive definite function on* $\hat{K}$ *for each* $x \in K$.

Proof. For $\lambda_1,\dots,\lambda_n \in \mathbb{C}$, $\alpha_1,\dots,\alpha_n \in \hat{K}$ denote by $\lambda = \sum_{i=1}^n \lambda_i p_{\alpha_i}$. Then $(\lambda * \lambda^*)\check{}(x) = |\sum_{i=1}^n \lambda_i \alpha_i(x)|^2 \geqslant 0$. Hence for $x \in K$

$$\sum_{i=1}^n \sum_{j=1}^n \lambda_i \overline{\lambda_j}\, p_{\alpha_i} * p_{\overline{\alpha_j}}(\widehat{p_x}\varphi) = \int \widehat{p_x}\varphi(\alpha)\, d\lambda * \lambda^*(\alpha)$$
$$= [\varphi(\lambda * \lambda^*)]\check{}(\bar{x}) \geqslant 0$$

by Proposition 1(a).

We are not able to give a complete answer to the question whether the converse implication of Proposition 3 is valid. But we can prove the following statement:

THEOREM 2. *Assume that K is a compact hypergroup such that $\hat{K}$ is a hypergroup with respect to pointwise multiplication. If φ is a bounded function such that $\widehat{p_x}\varphi$ is positive definite for each $x \in K$, then there exists a unique positive measure $\nu \in M(K)$ such that $\varphi = \hat{\nu}$. Conversely, $\widehat{p_x}\hat{\nu}$ is positive definite for each $\nu \in M^+(K)$* and $x \in K$.

Proof. Of course $\widehat{p_x}\hat{\nu} = (p_x * \nu)\hat{}$ is a positive definite function, if $\nu \in M^+(K)$. The proof of the converse implication is motivated by [7], see also [15, Theorem 30.2]. Denote by $T(K)$ the linear span of $\hat{K}$. Define the linear functional ϕ on $T(K)$ by

$$\phi(f) = \sum_{i=1}^n \lambda_i \varphi(\alpha_i)$$

for $f = \sum_{i=1}^n \lambda_i \alpha_i$, $\lambda_i \in \mathbb{C}$, $\alpha_i \in \hat{K}$. Now ϕ is well defined, since $\hat{K}$ is a linear independent set in $T(K)$. We state that $p_x * (f\bar{f}) \in T(K)$ and

$$\phi(p_x * (f\bar{f})) \geqslant 0 \tag{1}$$

for each $f \in T(K)$, $x \in K$. In fact if $f = \sum_{i=1}^n \lambda_i \alpha_i \in T(K)$, we have $\alpha_i \overline{\alpha_j} = \sum_{k=1}^{m_{ij}} b_k^{ij} \beta_k^{ij}$, $\beta_k^{ij} \in \hat{K}$, $b_k^{ij} \geqslant 0$. Observe that $\hat{K}$ is a discrete hypergroup. Hence

$$p_x * (f\bar{f}) = \sum_{i,j=1}^n \lambda_i \overline{\lambda_j} \sum_{k=1}^{m_{ij}} b_k^{ij} \cdot \beta_k^{ij}(\bar{x})\, \beta_k^{ij} \in T(K).$$

Further

$$\phi(p_x * (f\bar{f})) = \sum_{i,j=1}^n \lambda_i \overline{\lambda_j} \sum_{k=1}^{m_{ij}} b_k^{ij} \beta_k^{ij}(\bar{x})\, \varphi(\beta_k^{ij})$$
$$= \sum_{i,j=1}^n \lambda_i \overline{\lambda_j}\, p_{\alpha_i} * p_{\overline{\alpha_j}}(\widehat{p_x}\varphi) \geqslant 0,$$

since $\widehat{p_x}\varphi$ is positive definite. Now we prove in a similar way as in [15, (30.2)] that

$$\phi(g) \geqslant 0 \tag{2}$$

for $g \in T(K)$, $g \geqslant 0$. For $f = \sum_{i=1}^n \lambda_i \alpha_i \in T(K)$ define $f_\phi \in T(K)$ by

$$f_\phi(x) = \phi(p_x * f) = \sum \lambda_i \varphi(\alpha_i)\, \overline{\alpha_i(x)}.$$

Given $h \in T(K)$, $h \geqslant 0$ denote by ϕ^h the linear functional on $T(K)$ defined by

$$\phi^h(f) = \int h(x) f_\phi(x)\, dm(x).$$

Since $(f\bar{f})_\phi(x) = \phi(p_x * (f\bar{f})) \geqslant 0$ for each $x \in K$,

$$\phi^h(f\bar{f}) \geqslant 0 \tag{3}$$

holds for each $f \in T(K)$. If $h = \sum_{i=1}^n \lambda_i \alpha_i$, $\lambda_i \neq 0$, denote

$$c_i = \int |\alpha_i(x)|^2\, dm(x) \text{ and } M = \sup_{x \in K} \left| \sum_{i=1}^n c_i^{-1}\, \phi^h(\alpha_i)\, \alpha_i(x) \right|.$$

Using [27, Theorem II.2.2] one obtains

$$|\phi^h(f)| = \left| \sum_{i=1}^n \left(c_i^{-1} \int f(x)\, \overline{\alpha_i(x)}\, dm(x)\right) \phi^h(\alpha_i) \right| \leqslant M \, \|f\|_u.$$

The norm-continuity of ϕ^h, [29, Theorem 2.13] and (3) imply that

$$\phi^h(g) \geqslant 0 \tag{4}$$

for $g \in T(K)$, $g \geqslant 0$. To prove (2) now choose an approximate unit (h_κ) in $L^1(K)$, $h_\kappa \in T(K)$, $h_\kappa \geqslant 0, \|h_\kappa\|_1 = 1$ according to [29, Lemma 2.12]. For $\alpha \in \hat{K}$ we have $\widehat{h_\kappa}(\alpha) \to 1$. Hence for $g \in T(K)$, $g \geqslant 0$, $\phi^{h_\kappa}(g)$ tends to $\phi(g)$. Thus $\phi(g) \geqslant 0$ by means of (4). Using (2) one proves as in [15, (30.2), p. 157] that ϕ is norm continuous. $T(K)$ is norm dense in $C(K)$. Hence there exists a unique positive measure $\nu \in M(K)$ such that

$$\phi(g) = \int g\, d\nu \qquad \text{for} \quad g \in T(K).$$

In particular $\varphi(\alpha) = \phi(\alpha) = \hat{\nu}(\alpha)$ for every $\alpha \in \hat{K}$.

COROLLARY 4. *Assume that K is a compact hypergroup such that $\hat{K}$ is a hypergroup with respect to pointwise multiplication. Then $\varphi \in C(\hat{K})$ is strongly positive definite if and only if $p_x \varphi$ is a bounded positive definite function for each $x \in K$.*

There is a (rather weak) duality theorem for a hypergroup K, whose dual $\hat{K}$ is a hypergroup with respect to pointwise multiplication: $K \subseteq (\hat{K})\hat{}$, see [17, 12.4]. If in addition $K = (\hat{K})\hat{}$ holds, we shall call K a strong hypergroup. Examples for strong hypergroups are given in 1a and Section 2, Example 2a.

PROPOSITION 4. *Assume that $\hat{K}$ is a hypergroup with respect to pointwise multiplication. K is a strong hypergroup if and only if each bounded positive definite $\varphi \in C(\hat{K})$ is strongly positive definite.*

Proof. If K is a strong hypergroup, [17, Theorem 12.3B] yields that every bounded positive definite function $\varphi \in C(\hat{K})$ is strongly positive definite. Conversely, let $\varphi \in (\hat{K})\hat{}$. Hence φ is a bounded positive definite function. By the assumption and Theorem 1 we can write $\varphi = \hat{\nu}$, where $\nu \in M^+(K)$. On the other hand the linear functional on $C_{00}(\hat{K})\check{}$

$$\phi_\varphi(\check{f}) = \int \varphi(\alpha) f(\alpha)\, d\pi(\alpha), \qquad f \in C_{00}(\hat{K}),$$

is multiplicative. Since $\phi_\varphi(\check{f}) = \int \check{f}(\bar{x})\, d\nu(x)$, we see that ϕ_φ is norm continuous. Using Proposition 2, ϕ_φ admits a unique extension to $C_0(K)$. This extension is multiplicative, too. Hence there exists a point $x \in K$ such that

$$\phi_\varphi(\check{f}) = \check{f}(x),$$

for each $f \in C_{00}(\hat{K})$, i.e., $\varphi = \widehat{p_x}$. Therefore $K = (\hat{K})\hat{}$, see [17, 12.4].

Finally we present a consequence of [17, Theorem 12.3B], which has a nice interpretation for our examples in Section 2.

COROLLARY 5. *Let $\varphi \in L^1(K)$ be a bounded positive definite function. Then $\hat{\varphi}$ is nonnegative, $\hat{\varphi} \in L^1(\hat{K})$, and $(\hat{\varphi})\check{} = \varphi$.*

Proof. Reference [17, Theorem 12.3B and Lemma 12.2B] yields that $\hat{\varphi} \in L^1(\hat{K})$ and $\hat{\varphi} \geqslant 0$. The inversion theorem, [17, Theorem 12.2C] says that $(\hat{\varphi})\check{} = \varphi$.

EXAMPLE 1a. First we describe a rather general class of strong hypergroups. Let G denote a locally compact group and let B denote a subgroup of the automorphism group $\mathrm{Aut}(G)$ that contains the group $I(G)$ of inner automorphisms. If the closure $\bar{B}$ of B in $\mathrm{Aut}(G)$ is compact, then the $\bar{B}$-orbit space G_B of G is a commutative hypergroup with natural operations, see [23, Sect. 1]. In [14] it is shown that $\widehat{G_B}$ is a hypergroup with respect to pointwise multiplication. That $(\widehat{G_B})\hat{}$ is equal to G_B follows by [13]. This class includes among others the set of conjugacy classes of compact groups or $K = (\mathbb{R}^n)_{SO(n)}$.

EXAMPLE 1b. Theorem 1 applies to any commutative double coset space $G//H$, where H is a compact subgroup of the locally compact group G. For a concrete example take $G = SL(2, \mathbb{C})$ and $H = SU(2)$. Then $[0, \infty[$ is a model for $K = G//H$, see [17, 15.5 and 9.5], and $\hat{K} \cong [-1, \infty[$, whereas $\operatorname{supp} \pi \cong [0, \infty[$.

2. Applications to Orthogonal Polynomial Expansions

In the sequel K is $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ bearing a hypergroup structure corresponding to certain orthogonal polynomial sequences, see [19]. We have to set up some notation. Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ be three real-valued sequences such that $a_n > 0$, $c_n > 0$, $b_n \geqslant 0$, and $a_n + b_n + c_n = 1$. If (a_n), (b_n), (c_n) satisfy a certain positivity property (P), see [19, Sect. 2], these sequences determine a hypergroup structure on $\mathbb{N}_0$. The "generating" convolution is given by

$$p_1 * p_n = a_n p_{n+1} + b_n p_n + c_n p_{n-1}, \qquad n \in \mathbb{N}.$$

The Haar measure on $\mathbb{N}_0$ is given (up to normalization) by

$$h(0) = 1, \quad h(1) = 1/c_1, \quad h(n) = \prod_{k=1}^{n-1} a_k \Big/ \prod_{k=1}^{n} c_k, \qquad n = 2, 3, \ldots.$$

Fixing $a_0 > 0, b_0 \in \mathbb{R}$ such that $a_0 + b_0 = 1$ define

$$P_0(x) = 1, \qquad P_1(x) = (1/a_0)\, x - (b_0/a_0)$$

$$P_{n+1}(x) = (1/a_n)\, P_1(x)\, P_n(x) - (b_n/a_n)\, P_n(x) - (c_n/a_n)\, P_{n-1}(x), \qquad n \in \mathbb{N}. \qquad \text{(R)}$$

Now $(P_n(x))$ is an orthogonal polynomial sequence. For $x \in \mathbb{R}$ denote $\alpha_x : \mathbb{N}_0 \to \mathbb{R}$, $\alpha_x(n) = P_n(x)$, and let $D_S = \{x \in \mathbb{R} : (P_n(x)) \text{ is bounded}\}$. Then $\widehat{\mathbb{N}_0} = \{\alpha_x : x \in D_S\}$, D_S is homeomorphic to $\widehat{\mathbb{N}_0}$, and $D_S \subseteq [1 - 2a_0, 1]$. The Plancherel measure π is the orthogonalization measure of $(P_n(x))$. If $D_S = \widehat{\mathbb{N}_0}$ is a hypergroup with respect to pointwise multiplication, then $\mathbb{N}_0$ is a strong hypergroup, [19, Proposition 2]. Likewise D_S is then a strong hypergroup.

THEOREM 1′. *Assume that* $(a_n), (b_n), (c_n)$ *have property* (P). *Let* $\varphi \in C(D_s)$ *be a continuous function on* D_S. *Then*

$$\varphi(x) = \sum_{n=0}^{\infty} d_n P_n(x)\, h(n)$$

holds for $x \in \operatorname{supp} \pi$, where $d_n \geqslant 0$, $\sum_{n=0}^{\infty} d_n h(n) < \infty$, *provided*

$$\int_{D_S} \varphi(x)\, d\mu(x) \geqslant 0 \text{ for any measure } \mu \in M(D_S) \text{ with}$$

$$\int_{D_S} P_n(x)\, d\mu(x) \geqslant 0.$$

The coefficients d_n *are given by* $d_n = \int \varphi(x) P_n(x)\, d\pi(x)$. *Conversely for any sequence* $(d_n)_{n \in \mathbb{N}_0}$, $d_n \geqslant 0$, $\sum d_n h(n) < \infty$ *and* $\varphi(x) = \sum d_n P_n(x)\, h(n)$, $x \in D_S$, *the inequality* $\int \varphi(x)\, d\mu(x) \geqslant 0$ *holds for* $\mu \in M(D_S)$ *with* $\int P_n(x)\, d\mu(x) \geqslant 0$.

We call $(g_k)_{k \in \mathbb{N}}$, $g_k \in C(D_S)$ a *positive definite approximate unit for* D_S, if $\| g_k \|_1 \leqslant 1$, $\check{g}_k(n) \geqslant 0$ and $\check{g}_k(n) \to 1$ as $k \to \infty$ for each $n \in \mathbb{N}$. If D_S admits such an approximate unit we can give an improvement of Theorem 1′.

LEMMA 1. *Let* $(a_n), (b_n), (c_n)$ *have property* (P). *Assume that* D_S *admits a positive definite approximate unit. Let* $\varphi \in C(D_S)$ *such that* $d_n = \int \varphi(x) P_n(x)\, d\pi(x) \geqslant 0$ *holds. Then* $\sum d_n h(n) < \infty$.

Proof. Let E be a finite subset of $\mathbb{N}_0$. Denote the positive definite approximate unit by (g_k). Then [17, Theorem 12.11] yields that

$$\sum_{n \in E} \check{g}_k(n)\, d_n h(n) \leqslant \sum_{n \in \mathbb{N}_0} \check{g}_k(n)\, d_n h(n) = \int g_k(x)\, \varphi(x)\, d\pi(x)$$
$$\leqslant \| g_k \|_1 \|\varphi\|_u \leqslant \|\varphi\|_u$$

for each $k \in \mathbb{N}$. Hence $\sum_{n \in E} d_n h(n) \leqslant \|\varphi\|_u$ and then $\sum_{n \in \mathbb{N}_0} d_n h(n) \leqslant \|\varphi\|_u$.

THEOREM 3. *Let* $(a_n), (b_n), (c_n)$ *have property* (P). *Assume that* D_S *admits a positive definite approximate unit. Let* $\varphi \in C(D_S)$. *Then* $\varphi(x) = \sum d_n P_n(x)\, h(n)$ *holds for each* $x \in \operatorname{supp} \pi$, *where* $d_n \geqslant 0$, $\sum d_n h(n) < \infty$, *provided*

$$\int \varphi(x) P_n(x)\, d\pi(x) \geqslant 0 \qquad \text{for each} \quad n \in \mathbb{N}_0.$$

The coefficients d_n *are given by* $d_n = \int \varphi(x) P_n(x)\, d\pi(x)$.

Proof. Denote $d_n = \check{\varphi}(n)$. By Lemma 1 $\sum_{n=0}^{\infty} d_n P_n(x)\, h(n)$ is uniformly convergent, say to $\psi(x)$. Since $\check{\psi} = \check{\varphi}$ we have $\varphi(x) = \psi(x)$ for each $x \in \operatorname{supp} \pi$.

THEOREM 4. *Let (a_n), (b_n), (c_n) have property (P) and assume that D_S is a hypergroup with respect to pointwise multiplication. Let $\varphi \in C(D_S)$. The following are equivalent*:

(a) $\varphi(x) = \sum d_n P_n(x)\, h(n)$ *for each* $x \in D_S$, *where* $d_n \geqslant 0$, $\sum d_n h(n) < \infty$;

(b) $\int \varphi(x)\, d\mu(x) \geqslant 0$ *for each* $\mu \in M(D_S)$ *with* $\int P_n(x)\, d\mu(x) \geqslant 0$;

(c) $d_n = \int \varphi(x)\, P_n(x)\, d\pi(x) \geqslant 0$; *and*

(d) *for any n-tuple* $x_1, ..., x_n \in D_S$ *the matrix* $(p_{x_i} * p_{x_j}(\varphi))_{1 \leqslant i,j \leqslant n}$ *is positive definite.*

Proof. Since D_S is a dual hypergroup, we have $\operatorname{supp} \pi = D_S$. By the preceding and [17, Theorem 12.3B] it remains to show that D_S admits a positive definite approximate unit. According to [9, Theorem 2.8] choose an approximate unit (f_k) in $L^1(D_S)$ such that $f_k \in C(D_S)$. Define $g_k = f_k * f_k^* / \| f_k * f_k^* \|_1$. One can easily establish that (g_k) is a positive definite approximate unit for D_S.

Now we point out how the positivity of connection coefficients can be used in the discussion of positive definiteness. If $(Q_n(x))$ and $(P_n(x))$ are two sequences of orthogonal polynomials, ($Q_n(x)$, $P_n(x)$ of degree n), then one can write

$$Q_n(x) = \sum_{k=0}^{n} c_{k,n} P_k(x).$$

The numbers $c_{k,n}$ are called connection coefficients. The problem to determine these coefficients or to decide when these coefficients are nonnegative is thoroughly studied by Askey and others, see [1, Lecture 7]. We shall shortly write $(P_n(x)) \geqslant (Q_n(x))$, if every $c_{k,n}$ is nonnegative.

Let (a_n), (b_n), (c_n) (resp. (a_n'), (b_n'), (c_n')) have the property (P). Fix a_0, b_0 (resp. a_0', b_0') and define $(P_n(x))$, h, D_S, π (resp. $(Q_n(x))$, h', D_S', π') as above. We assume that $D_S = D_S'$. We note that $\operatorname{supp} \pi$ is infinite. This is implied, for example, by Proposition 2. The following result can be used in the discussion of a problem in numerical analysis (cf. [22]).

THEOREM 5. *With the preceding notation the following are equivalent*:

(a) $(P_n(x)) \geqslant (Q_n(x))$,

(b) $\mathrm{PM}(D_S) \subseteq \mathrm{PM}(D_S')$,

(c) $\mathrm{SP}(D_S) \supseteq \mathrm{SP}(D_S')$.

If in addition $\operatorname{supp} \pi' = D'_S$ *holds, then each of these conditions is equivalent to the following*:

(d) *If* $\varphi \in C(D_S)$ *with* $d'_n = \int \varphi(x)\, Q_n(x)\, d\pi'(x) \geqslant 0$, $\sum d'_n h'(n) < \infty$, *then* $d_n = \int \varphi(x)\, P_n(x)\, d\pi(x) \geqslant 0$, $\sum d_n h(n) < \infty$.

Proof. Obviously, (a) → (b), and (b) → (c). Since $Q_m \in \mathrm{SP}(D'_S)$, condition (c) and Theorem 1 yield that

$$Q_m(x) = \sum_{k=0}^{m} c_{k,m} P_k(x) \text{ for each } x \in \operatorname{supp} \pi, \qquad \text{where } c_{k,m} \geqslant 0.$$

But $\operatorname{supp} \pi$ is infinite. Hence $(P_n(x)) \geqslant (Q_n(x))$. Suppose that $\operatorname{supp} \pi' = D'_S$. Then $\varphi(x) = \sum d'_n\, Q_n(x)\, h'(n)$ for each $x \in D'_S$. In particular, $\varphi \in \mathrm{SP}(D'_S)$. Hence condition (c) → (d). Conversely, by (d) we see that $c_{n,m}/h(n) = \int Q_m(x)\, P_n(x)\, d\pi(x) \geqslant 0$.

Applying Corollary 5 we have a contribution to positive sums of orthogonal polynomials, see [1, Lectures 1, and 8].

COROLLARY 5′. *Let* (a_n), (b_n), (c_n) *have property* (P). *Let* $d = (d_n)_{n \in \mathbb{N}_0}$ *be a sequence such that* $\sum |d_n| h(n) < \infty$. *Then* $\sum d_n\, P_n(x)\, h(n) \geqslant 0$ *for each* $x \in D_S$ *if and only if* $(p_{m_i} * p_{m_j}(d))_{1 \leqslant i,j \leqslant n}$ *are positive definite matrices for each n-tuple* $m_1, \ldots, m_n \in \mathbb{N}_0$.

EXAMPLE 2a (Jacobi polynomials). Fix $\alpha, \beta \in \mathbb{R}, \alpha \geqslant \beta > -1, \alpha + \beta + 1 \geqslant 0$. Define for $n \in \mathbb{N}$:

$$a_n = \frac{2(n+\alpha+\beta+1)(n+\alpha+1)(\alpha+\beta+2)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)\, 2(\alpha+1)},$$

$$b_n = \frac{\alpha-\beta}{2(\alpha+1)} \left[1 - \frac{(\alpha+\beta+2)(\alpha+\beta)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta)}\right],$$

$$c_n = \frac{2n(n+\beta)(\alpha+\beta+2)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)\, 2(\alpha+1)}.$$

Let $a_0 = 2(\alpha+1)/(\alpha+\beta+2)$, $b_0 = (\beta-\alpha)/(\alpha+\beta+2)$. Then (R) defines $P_n(x) = P_n^{(\alpha,\beta)}(x)$, the classical Jacobi polynomials. The sequences (a_n), (b_n), (c_n) have the property (P) and $\operatorname{supp} \pi = D_S = [-1, 1]$, see [19, Sect. 3(*a*)]. If in addition $\beta \geqslant -\frac{1}{2}$ or $\alpha + \beta \geqslant 0$, then $D_S = [-1, 1]$ is a hypergroup with respect to pointwise multiplication. Moreover, $\mathbb{N}_0$ and $[-1, 1]$ are strong hypergroups, see [19, Sect. 4, ad(a)]. For $\alpha = \beta$ we have the ultraspherical polynomials.

EXAMPLE 2b (Continuous q-Jacobi polynomials). Fix $\alpha,\ \beta,\ q \in \mathbb{R}$, $\alpha \geqslant \beta > -1$, $\alpha+\beta+1 \geqslant 0$, $0<q<1$. Let

$$A_n = \frac{(1-q^{\alpha+n+1})(1+q^{\beta+n+1})(1+q^{n+1})(1-q^{\alpha+\beta+n+1})}{\sqrt{q}(1-q^{\alpha+\beta+2n+1})(1-q^{\alpha+\beta+2n+2})}$$

$$C_n = \frac{\sqrt{q}(1-q^n)(1+q^{\alpha+\beta+n})(1+q^{\alpha+n})(1-q^{\beta+n})}{(1-q^{\alpha+\beta+2n})(1-q^{\alpha+\beta+2n+1})}.$$

Consider $Q_n^{(\alpha,\beta)}(x;q) = Q_n(x)$ defined by

$$2xQ_n(x) = A_n Q_{n+1}(x) - (A_n + C_n - \sqrt{q} - 1/\sqrt{q})\, Q_n(x) + C_n Q_{n-1}(x),$$
$$Q_0(x) = 1, \qquad Q_{-1}(x) = 0.$$

The $Q_n(x)$ are (up to normalization) the continuous q-Jacobi polynomials studied in [12, 21], see also [3, (3.4)]. Observe that the zeros of $Q_n(x)$ are contained in $]-1, 1[$, see [12, Sect. 2]. Hence $\gamma_n = Q_n(1) > 0$. Let

$$a_0 = \frac{(1-q^{\alpha+1})(1+q^{\beta+1})}{(1-q^{\alpha+\beta+2})}, \qquad b_0 = \frac{q^{\alpha+1} - q^{\beta+1}}{(1-q^{\alpha+\beta+2})}.$$

Now define

$$a_n = \frac{A_n \gamma_{n+1}}{2a_0 \gamma_n}, \qquad b_n = \frac{\sqrt{q} + 1/\sqrt{q} - A_n - C_n}{2a_0} - \frac{b_0}{a_0}, \quad c_n = \frac{C_n \gamma_{n-1}}{2a_0 \gamma_n}.$$

For the polynomials $P_n^{(\alpha,\beta)}(x;q)$, which are defined by (R), we have $P_n^{(\alpha,\beta)}(x;q) = Q_n^{(\alpha,\beta)}(x;q)/\gamma_n$. Hence $a_n + b_n + c_n = 1$. Now use [21] or [12, Theorem 1]. In particular we see that $b_n \geqslant 0$. But the main contribution of [21] is that $(a_n), (b_n), (c_n)$ satisfy the property (P). Further $\operatorname{supp} \pi = [-1, 1]$, see [3, (4.1)], and $[-1, 1] \subseteq D_S \subseteq [1 - 2a_0, 1]$. The limit case $q \to 1$ yields Example 2a.

EXAMPLE 2c (Continuous q-ultraspherical polynomials). Fix $-1 < \beta < 1$, $0 < q < 1$. The continuous q-ultraspherical polynomials $C_n(x; \beta | q)$ correspond in the described way to a hypergroup structure on $\mathbb{N}_0$. For details we cite [19, Sect. 3(c)]. Further $\operatorname{supp} \pi = D_S = [-1, 1]$ holds. These examples are partially contained in Example 2b.

EXAMPLE 2d (Associated continuous q-ultraspherical polynomials). Fix $0 < \beta \leqslant q < 1, 0 < \alpha < 1$. The associated q-ultraspherical polynomials $C_n^\alpha(x; \beta | q)$ bear a hypergroup structure, see [8] or [20, 3(f)]. We have $\operatorname{supp} \pi = D_S = [-1, 1]$.

EXAMPLE 2e (Associated Legendre polynomials). Fix $\nu \geqslant 0$. The associated Legendre polynomials $P_n(x;\nu)$ correspond to a hypergroup structure on $\mathbb{N}_0$, see [19, Sect. 3(b)]. We have supp $\pi = D_S = [-1, 1]$.

EXAMPLE 2f (Polynomials connected with homogeneous trees). Fix $a \geqslant 2$. Define $a_n = (a-1)/a$, $b_n = 0$, $c_n = 1/a$, $n \in \mathbb{N}$, $a_0 = 1$, $b_0 = 0$. In this way there is defined a hypergroup structure on $\mathbb{N}_0$, [19, Sect. 3(d)], the orthogonal polynomial sequence being intimately connected with homogeneous trees. We have $D_S = [-1, 1]$, but supp $\pi = [-2\sqrt{a-1}/a, 2\sqrt{a-1}/a]$, see [10, Théorème 1]. In particular $\mathbb{N}_0$ is not a strong hypergroup, if $a > 2$.

EXAMPLE 2g (Generalized Tchebichef polynomials). Let $\alpha, \beta \in \mathbb{R}$, $\beta > -1$, $\alpha \geqslant \beta + 1$. The generalized Tchebichef polynomials $T_n^{(\alpha,\beta)}(x)$ bear a hypergroup structure [19, 3(f)]. We have supp $\pi = D_S = [-1, 1]$. If $\beta > -\frac{1}{2}$, then D_S is not a dual hypergroup, see [19, Sect. 4, ad(f)].

Further examples may be found in [19]. Concerning applications of Theorem 5 to concrete examples we refer to [1, (7.33), (7.34)] saying that $(P_n^{(\alpha,\alpha)}(x)) \geqslant (P_n^{(\nu,\nu)}(x))$ if $\gamma \geqslant \alpha > -1$ and $(P_n^{(\alpha,\beta)}(x)) \geqslant (P_n^{(\nu,\beta)}(x))$ if $\gamma \geqslant \alpha > -1$ and to [3, (4.15)] saying that

$$(C_n(x;\beta|q)) \geqslant (C_n(x;\gamma|q) \quad \text{if} \quad 0 < q < 1, \quad -1 < \gamma \leqslant \beta < 1.$$

The general result of [2, Theorem 1] yields that

$$(P_n^{(0,0)}(x)) \geqslant (P_n(x;\nu)) \geqslant (P_n(x;\mu)) \geqslant (P_n^{(1/2,\ 1/2)}(x)) \qquad \text{if} \quad \mu \geqslant \nu \geqslant 0,$$

where $P_n(x, \nu)$ are the associated Legendre polynomials of Example 2e.

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